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Nonuniform Right Definiteness

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The multiparameter eigenvalue problem

$$W_m(\boldsymbol{\lambda})\, x_m = x_m, \qquad W_m(\boldsymbol{\lambda}) = T_m + \sum_{n=1}^{k} \lambda_n V_{mn}, \qquad m = 1,..., k,$$

where $\boldsymbol{\lambda} \in \mathbb{C}^k$, x_m is a nonzero element of the separable Hilbert space H_m, and T_m and V_{mn} are compact symmetric is studied. Various properties, including existence and uniqueness, of $\boldsymbol{\lambda} = \boldsymbol{\lambda}^{\mathbf{i}} \in \mathbb{C}^k$ for which the i_mth greatest eigenvalue of $W_m(\boldsymbol{\lambda}^{\mathbf{i}})$ equals one are proved. "Right definiteness" is assumed, which means positivity of the determinant with (m, n)th entry $(y_m, V_{mn} y_m)$ for all nonzero $y_m \in H_m$, $m = 1 \cdots k$. This gives a "Klein oscillation theorem" for systems of an o.d.e. satisfying a definiteness condition that is usefully weaker than in previous such results. An expansion theorem in terms of the corresponding eigenvectors $x_m^{\mathbf{i}}$ is also given, thereby connecting the abstract oscillation theory with a result of Atkinson.

1. INTRODUCTION

Let T'_m and V'_{mn} be self-adjoint operators on separable Hilbert spaces H_m, with V'_{mn} bounded and T'_m bounded below with compact resolvent. Thus for some real α,

$$S_m = T'_m + \alpha I_m \tag{1.1}$$

has a positive compact inverse, I_m being the identity on H_m. The use of a prime is to distinguish this situation from that of the abstract.

The multiparameter eigenvalue problem

$$W'_m(\boldsymbol{\lambda})\, x'_m = 0, \qquad W'_m(\boldsymbol{\lambda}) = T'_m + \sum_{n=1}^{k} \lambda_n V'_{mn} \tag{1.2}$$

for nonzero $x'_m \in H_m$ has been studied in many works, mostly under the condition

$$|\det V'(u)| > \beta \tag{1.3}$$

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for some $\beta > 0$ for all unit $u_m \in H_m$. Here we write $V'(u)$ for the matrix with (m, n)th entry $(u_m, V'_{mn} u_m)$. Problems of the form (1.2), (1.3) arise with certain systems of an o.d.e. in t, where

$$V'_{mn} u_m(t) = a_{mn}(t)\, u_m(t)$$

and $|\det A(t)| \geqslant \gamma$ for some $\gamma > \beta$ and for all t. In some applications, $\gamma = 0$ yet (1.3) still holds with $\beta = 0$. In particular, all the systems in [1, Chap. 1] arising from separation of the wave equation into different coordinates can be put into forms with bounded V'_{mn} satisfying (1.3) for $\beta = 0$ but *not* for $\beta > 0$.

The connection with the problem described in the abstract is as follows. The argument is essentially as for [4, Theorem 2.2] which treats the case where $\alpha = 0$ in (1.1). Obviously (1.2) is equivalent to

$$\left(\alpha I_m - \sum_{n=1}^{k} \lambda_n V'_{mn}\right) x'_m = S_m x'_m \tag{1.4}$$

which may be shown (under the stated conditions on T'_m and V'_{mn}) to be equivalent to

$$W_m(\boldsymbol{\lambda})\, x_m = x_m, \qquad W_m(\boldsymbol{\lambda}) = T_m + \sum_{n=1}^{k} \lambda_n V_{mn} \tag{1.5}$$

with $x_m = S_m^{1/2} x'_m$, where $T_m = \alpha S_m^{-1}$ and $V_{mn} = -S_m^{-1/2} V'_{mn} S_m^{-1/2}$ are compact symmetric. Moreover if $\boldsymbol{\lambda} \in \mathbb{R}^k$ then the i_mth least eigenvalue (counted according to multiplicity) of $W'_m(\boldsymbol{\lambda})$ vanishes if and only if the i_mth greatest eigenvalue of $W_m(\boldsymbol{\lambda})$ equals one. Finally (1.3) with $\beta = 0$ obviously implies

$$|\det V(u)| > 0, \tag{1.6}$$

where $V(u)$ has (m, n)th entry $(u_m, V_{mn} u_m)$.

In summary, we lose nothing by solving (1.5), (1.6) instead of (1.2), (1.3). In fact we gain something since not all (e.g., integral equation) cases of (1.5) correspond to (e.g., differential) Eqs. (1.2), assuming only (as we shall from now on) that T_m and V_{mn} are compact symmetric. In Section 2 we give some standard properties of eigentuples indexed by $\mathbf{i} = (i_1, ..., i_k)$, where i_m is as used above. In particular, at most one $\boldsymbol{\lambda}$ corresponds to a given $\mathbf{i}$. The main novelty is the boundedness principle of Section 3. This allows us to dispense with the uniformity in (1.3) when $\beta > 0$, a necessary ingredient of previous "oscillation theorems" for (1.2), e.g., [5, Theorem 2]. We deduce existence of $\boldsymbol{\lambda} = \boldsymbol{\lambda}^1$ from finite dimensional approximations (where $\beta > 0$ may be assumed).

Our final Section 4 treats the expansion question for the corresponding eigenvectors, the results being very similar to those of Atkinson [2, Chap. 11]. For the case $\alpha = 0$ in (1.1), our result sharpens [2, Sect. 11.10] in a sense. We remark that if $\beta > 0$ holds in (1.3), then we may assume $\alpha = 0$ after an eigenvalue translation [3, Lemma 2.1]. In general our result overlaps with [2, Theorem 11.8.1], and a more detailed comparison will be given in Section 4.

2. Basic Properties of Eigentuples

Let

$$U_m = \{u_m \in H_m : \|u_m\| = 1\}, \qquad U = \mathop{\times}_{m=1}^{k} U_m,$$

and for each $u \in U$ set

$$t_m(u) = (u_m, T_m u_m), \qquad \mathbf{t}(u) = (t_1(u), \ldots, t_k(u)).$$

Similar definitions are made for $v_{mn}(u)$, $\mathbf{v}_m(u)$, $w_{mn}(\boldsymbol{\lambda}, u)$, and $\mathbf{w}(\boldsymbol{\lambda}, u)$. In particular,

$$\mathbf{w}(\boldsymbol{\lambda}, u) = \mathbf{t}(u) + V(u)\boldsymbol{\lambda},$$

where $V(u)$ comes from (1.6), and the other terms are viewed as column vectors.

We enumerate the positive eigenvalues of $W_m(\boldsymbol{\lambda})$, according to multiplicity, as

$$\rho_m^0(\boldsymbol{\lambda}) \geqslant \rho_m^1(\boldsymbol{\lambda}) \geqslant \cdots, \tag{2.1}$$

and we recall the minimax characterization

$$\rho_m^i(\boldsymbol{\lambda}) = \min\{\max\{w_m(\boldsymbol{\lambda}, u) : u \perp y_j, u \in U\} : y_j \in U_m, 0 \leqslant j < i\}, \tag{2.2}$$

valid for those $\boldsymbol{\lambda} \in \mathbb{R}^k$ for which $\rho_m^i(\boldsymbol{\lambda}) > 0$. In the sequel, the symbol $\boldsymbol{\lambda}^{\mathbf{i}}$ will denote any solution of the equations

$$\rho_m^{i_m}(\boldsymbol{\lambda}^{\mathbf{i}}) = 1, \qquad m = 1 \cdots k. \tag{2.3}$$

Theorem 2.1. (i) *If* $\boldsymbol{\lambda} \in \mathbb{C}^k$ *satisfies* (1.5) *then* $\boldsymbol{\lambda} \in \mathbb{R}^k$, *and* $\boldsymbol{\lambda} = \boldsymbol{\lambda}^{\mathbf{i}}$ *for some* $\mathbf{i}$.

(ii) *At most one eigenvalue* $\boldsymbol{\lambda}^{\mathbf{i}}$ *corresponds to a given* $\mathbf{i}$.

Remark. Similar arguments to those of this section can be found in, say, [2, Chap. 11; and 5], but we include them for convenience and later use.

Proof. (i) From (1.5) we have

$$1 = w_m(\boldsymbol{\lambda}, x)$$

and

$$1 = (W_m(\boldsymbol{\lambda})\, x_m, x_m) = w_m(\bar{\boldsymbol{\lambda}}, x), \qquad m = 1 \cdots k.$$

Consequently $V(x)(\boldsymbol{\lambda} - \bar{\boldsymbol{\lambda}}) = \mathbf{0}$ and reality of $\boldsymbol{\lambda}$ follows from (1.6). Discreteness of the spectra of the $W_m(\boldsymbol{\lambda})$ then yields $\boldsymbol{\lambda} = \boldsymbol{\lambda}^{\mathbf{i}}$ for some $\mathbf{i}$.

(ii) Suppose $\rho_m^{i_m}(\boldsymbol{\mu}) = 1$, $m = 1 \cdots k$, in addition to (2.3), and let $u'_m \in U_m$ maximise $w_m(\boldsymbol{\mu}, u)$ for u_m orthogonal to the first i_m eigenvectors of $W_m(\boldsymbol{\lambda}^{\mathbf{i}})$, enumerated according to (2.1). By (2.2)

$$w_m(\boldsymbol{\lambda}^{\mathbf{i}}, u') \leqslant 1 \leqslant w_m(\boldsymbol{\mu}, u')$$

so

$$V(u')(\boldsymbol{\lambda}^{\mathbf{i}} - \boldsymbol{\mu}) \leqslant \mathbf{0}$$

in the componentwise order. Interchanging $\boldsymbol{\lambda}^{\mathbf{i}}$ and $\boldsymbol{\mu}$, we find $u'' \in U$ so that

$$V(u'')(\boldsymbol{\lambda}^{\mathbf{i}} - \boldsymbol{\mu}) \geqslant \mathbf{0}$$

and since the numerical range of $\sum_{n=1}^{k} (\lambda_n^{\mathbf{i}} - \mu_n)\, V_{mn}$ is convex for each m, there exists $u \in U$ so that

$$V(u)(\boldsymbol{\lambda}^{\mathbf{i}} - \boldsymbol{\mu}) = \mathbf{0}.$$

The result now follows from (1.6). Q.E.D.

We say that $\boldsymbol{\lambda}^{\mathbf{i}}$ has *multiplicity* $\prod_{m=1}^{k} \dim E_m$, where E_m is the eigenspace of $W_m(\boldsymbol{\lambda}^{\mathbf{i}})$ corresponding to eigenvalue 1. Also $x^i = (x_1^i, \ldots, x_k^i)$, $x_m^i \in H_m$, $i = 1, 2, \ldots$, are δ_0-*orthogonal* if the determinant $\delta_0(x^i, x^j)$ with (m, n)th entry $(x_m^i, V_{mn} x_m^j)$ vanishes when $i \neq j$. Let us isolate the finite dimensional case for subsequent use.

THEOREM 2.2. *Suppose each H_m is finite dimensional. Then a δ_0-orthogonal set of* $\prod_{m=1}^{k} \dim H_m$ *eigenvectors $u^{\mathbf{i}} \in U$ exists.*

Proof. In finite dimensions, (1.6) implies

$$|\det V(u)| > \beta$$

for some $\beta > 0$ and all $u \in U$. Thus [5, Theorem 4], with T_m replaced by $T_m - I_m$, gives the result. Q.E.D.

COROLLARY 2.3. (i) *Each $\boldsymbol{\lambda}^{\mathbf{i}}$ has finite multiplicity $k^{\mathbf{i}}$, and a δ_0-orthogonal set of $k^{\mathbf{i}}$ corresponding eigenvectors $u^{\mathbf{i}} \in U$ exists.*

(ii) *A δ_0-orthogonal set of eigenvectors $u^{\mathbf{i}}$ exists in U spanning all possible $x^{\mathbf{i}}$ satisfying* (1.5).

Proof. (i) follows directly from Theorem 2.2 and compactness of $W_m(\boldsymbol{\lambda}^{\mathbf{i}})$.

(ii) follows from (i) and the fact that $\boldsymbol{\lambda}^{\mathbf{i}} \neq \boldsymbol{\lambda}^{\mathbf{j}}$ implies $\delta_0(x^{\mathbf{i}}, x^{\mathbf{j}}) = 0$. The latter follows from an argument similar to that for Theorem 2.1(i). Q.E.D.

DEFINITION. A sequence of linear operators C^j is *completely convergent* to C *if* $C^j u^j \to Cu$ whenever $u^j \rightharpoonup u$. Obviously this implies $C^j \to C$ strongly, and we give two sufficient conditions.

LEMMA 2.4. *$C^j \to C$ completely provided either of the following hold*:

(i) *$C^j \to C$ uniformly and C is compact*;

(ii) *$C^j = A^j Q B^j$ where $A^j \to A$ and $B^j \to B$ strongly and $AQB = C$, with A^j and B^j bounded, B^j symmetric and Q compact.*

Proof. (i) Use the expansion $C^j u^j - Cu = (C^j - C)\, u^j + C(u^j - u)$.

(ii) B is symmetric and bounded so $(f, B^j u^j) - (f, Bu) = ((B^j - B) f, u^j) + (Bf, u^j - u) \to 0$ proves $B^j u^j \rightharpoonup Bu$. Thus $v^j = QB^j u^j \to QBu = v$ and the result now follows from $A^j v^j - Av = A^j(v^j - v) + (A^j - A)v$. Q.E.D.

Special cases of the following convergence theorem will be needed at various points. The results may be interpreted in terms of continuous dependence of (1.5) on parameters (cf. [5, Theorem 10]).

THEOREM 2.5. *Suppose T^j_m and V^j_{mn} are sequences of compact symmetric operators, completely convergent to T_m and V_{mn}, respectively, and let $(\boldsymbol{\lambda}^{\mathbf{i}(j)}, u^{\mathbf{i}(j)})$ form an eigenpair for* (1.5) *with $W_m(\boldsymbol{\lambda})$ replaced by $W^j_m(\boldsymbol{\lambda}) = T^j_m + \sum_{n=1}^k \lambda_n V^j_{mn}$.*

If $\boldsymbol{\lambda}^{\mathbf{i}(j)}$ *converges to a finite limit $\boldsymbol{\lambda}$ as $j \to \infty$ then $\boldsymbol{\lambda} = \boldsymbol{\lambda}^{\mathbf{i}}$ for* $\mathbf{i}$ *such that*

$$\liminf_{j\to\infty} \mathbf{i}(j) \leqslant \mathbf{i} \leqslant \limsup_{j\to\infty} \mathbf{i}(j)$$

componentwise.

If x_m is any weak limit point of $u_m^{\mathbf{i}(j)}$ as $j \to \infty$ then $x_m \in U_m$ and is also a strong limit point, and x is an eigenvector for (1.5) *corresponding to* $\boldsymbol{\lambda}^{\mathbf{i}}$.

Proof. By restricting attention to a subsequence, if necessary, we may assume $\mathbf{i}(j) \to \mathbf{i}$ as $j \to \infty$. Suppose, if possible, that

$$\rho_m^{i_m}(\boldsymbol{\lambda}) > 1.$$

Then an i_m-dimensional subspace $\Sigma_m \subset H_m$ exists such that

$$w_m(\boldsymbol{\lambda}, u) > 1 \qquad \text{for all } u \text{ with } u_m \in \Sigma_m \cap U_m.$$

It follows easily that $A_m^j := W_m^j(\boldsymbol{\lambda}^{\mathbf{i}(j)})$ has Rayleigh quotient greater than one on Σ_m, for large j. Thus the $i_m(j)$th eigenvalue of A_m^j exceeds one, and this contradiction to hypothesis forces

$$\rho_m^{i_m}(\boldsymbol{\lambda}) \leqslant 1. \tag{2.4}$$

Let y_m^{lj} be orthonormal eigenvectors of A_m^j corresponding to its $i_m(j)$ greatest eigenvalues α_m^{lj}, $1 \leqslant l \leqslant i_m(j)$, listed in decreasing order according to multiplicity. Let y_m^l be any weak limit point of the y_m^{lj} and let α_m^l be any limit point of the α_m^{lj}, as $j \to \infty$. Using subsequences if necessary, we see that $A_m^j \to W_m(\boldsymbol{\lambda})$ completely, so we have strong convergence on the left side of

$$A_m^j y_m^{lj} = \alpha_m^{lj} y_m^{lj}. \tag{2.5}$$

Since each $\alpha_m^{lj} \geqslant 1$, it follows that $\alpha_m^l \geqslant 1$ and $y_m^{lj} \to y_m^l$ strongly. Thus $W_m(\boldsymbol{\lambda})$ has at least $i_m(j)$ eigenvalues $\geqslant 1$, so (2.4) yields

$$\rho_m^{i_m(j)}(\boldsymbol{\lambda}) = 1,$$

i.e., $\boldsymbol{\lambda} = \boldsymbol{\lambda}^{\mathbf{i}}$. Moreover, if we fix $l = i_m(j)$ then we have $\alpha_m^{lj} = 1$ by hypothesis, and we may assume $y_m^{lj} = u_m^{\mathbf{i}(j)}$ without loss. Strong convergence of (2.5) then completes the argument. Q.E.D.

COROLLARY 2.6. *The eigenvalues $\boldsymbol{\lambda}^{\mathbf{i}}$ have no finite limit point.*

Proof. Choose eigenvectors $u^{\mathbf{i}}$ as in Corollary 2.3(ii). Suppose a subsequence $\boldsymbol{\lambda}^{\mathbf{i}(j)}$ of eigenvalues converges to $\boldsymbol{\lambda}$, and let x_m be a weak limit of $u_m^{\mathbf{i}(j)}$. By Theorem 2.5 with $W_m^j = W_m$, we have $x_m \in U$, whereas Corollary 2.3(ii) gives $\delta_0(x, x) = \det V(x) = 0$, contradicting (1.6). Q.E.D.

3. EXISTENCE OF EIGENTUPLES

We first formulate a boundedness principle.

THEOREM 3.1. *Suppose* (i) $\boldsymbol{\lambda}^j \in \mathbb{R}^k$ *and* $u^j \in U$ *satisfy*

$$\mathbf{w}(\boldsymbol{\lambda}^j, u^j) = \mathbf{1}, \qquad j = 1, 2, \ldots, \tag{3.1}$$

where $\mathbf{1}$ *is a column of ones, and* (ii) *there is a fixed finite dimensional subspace* $E_m \subseteq H_m$ *such that*

$$\mathbf{w}(\boldsymbol{\lambda}^j, u^0) \leqslant \mathbf{1} \tag{3.2}$$

for some j-dependent $u_m^0 \in U_m \cap E_m$. *Then the* $\boldsymbol{\lambda}^j$ *are bounded.*

Proof. Arguing by contradiction, we pass to a subsequence if necessary and assume $\|\boldsymbol{\lambda}^j\| \to \infty$ monotonically. Similarly, writing $\mathbf{v}_m^j = \mathbf{v}_m(u^j)$, we may assume that, for l and n belonging to complementary subsets of $\{1, 2,..., k\}$,

$$\mathbf{v}_l^j \to \mathbf{0} \tag{3.3}$$

while

$$\mathbf{v}_n^j \to \mathbf{v}_n \neq \mathbf{0}. \tag{3.4}$$

Write $\tilde{u}_l^j = u_l^0$, $\tilde{u}_n^j = u_n^j$ and define $\boldsymbol{\mu}^j$ by

$$V(\tilde{u}^j)\, \boldsymbol{\mu}^j = \mathbf{1} - \mathbf{t}(\tilde{u}^j). \tag{3.5}$$

By (3.4), compactness of the V_{mn} and finite dimensionality of the E_l, we may assume $\tilde{u}_m^j \to \tilde{u}_m \neq 0$, $m = 1 \cdots k$, by passing to a subsequence if necessary. Suppose the $\|\boldsymbol{\mu}^j\|$ are unbounded, so using another subsequence if necessary we may assume

$$\boldsymbol{\mu}^j/\|\boldsymbol{\mu}^j\| \to \hat{\boldsymbol{\mu}}, \qquad \|\hat{\boldsymbol{\mu}}\| = 1.$$

Thus (3.5) yields

$$V(\tilde{u})\hat{\boldsymbol{\mu}} = \mathbf{0}$$

and we contradict (1.5). It follows that the $\boldsymbol{\mu}^j$ are bounded, and we complete the proof by showing that $\boldsymbol{\lambda}^j = \boldsymbol{\mu}^j$ for large j.

We claim

$$\mathbf{v}_l^j \boldsymbol{\mu}^j \leqslant 1 - t_l(u^j)$$

for each l for large j. Indeed if not then $t_l(u^j) \geqslant \frac{1}{2}$ for infinitely many j, so the corresponding u_j^l possess a nonzero weak limit point. Thus (3.3), with any $u_m \in U_m$, $1 \leqslant m \neq l \leqslant k$, leads to a matrix $V(u)$ with zero lth row, contradicting (1.6). Further

$$\mathbf{v}_l^j \boldsymbol{\lambda}^j = 1 - t_l(u^j)$$

by (3.1), so

$$\mathbf{v}_l^j \mathbf{v}^j \geqslant 0, \qquad \mathbf{v}^j = \boldsymbol{\lambda}^j - \boldsymbol{\mu}^j.$$

Moreover (3.2) and (3.5) yield

$$\mathbf{v}_l^0 \mathbf{v}^j \leqslant 0.$$

As in the proof of Theorem 2.1(ii) we may find $u_l' \in U_l$ so that

$$\mathbf{v}_l(u')\, \mathbf{v}^j = 0,$$

where we have set $u_n' = u_n^j$. From (3.1) and (3.5) we have

$$\mathbf{v}_n(u')\, \mathbf{v}^j = 0,$$

so (1.6) yields $\mathbf{v}^j = \mathbf{0}$, i.e., $\boldsymbol{\lambda}^j = \boldsymbol{\mu}^j$. Q.E.D.

The existence result will be based on a sequence of orthoprojectors P_m^j on H_m strongly convergent to I_m. We fix any subspace E_m of H_m, of dimension greater than i_m, as $P_m^1 H_m$ and choose the subsequent P_m^j so that $P_m^j H_m \supseteq P_m^{j-1} H_m$ has finite dimension for $j = 2, 3, \ldots$, and $1 \leqslant m \leqslant k$. Now let $\boldsymbol{\lambda}^j$ be the solution of (2.3) but with the quadratic forms restricted to $P_m^j H_m$. More precisely, we replace the V_{mn} by the projected operators $V_{mn}^j = P_m^j V_{mn} P_m^j$ and similarly for the T_m. Restricted to the $P_m^j H_m$, the T_m^j and V_{mn}^j yield a finite dimensional version of (1.5) to which Lemma 2.2 applies. We take $\boldsymbol{\lambda}^j$ and u^j as the corresponding $\boldsymbol{\lambda}^{\mathrm{i}}$ and u^{i}, so (3.1) is immediate.

COROLLARY 3.3. *The $\boldsymbol{\lambda}^j$ converge to the unique eigenvalue $\boldsymbol{\lambda}^{\mathrm{i}}$ of (2.3) for (1.5). Moreover the weak limit points x_m of the u_m^j, $1 \leqslant m \leqslant k$, furnish corresponding eigenvectors $x \in U$.*

Proof. By the maximin version of (2.2), $w_m(\boldsymbol{\lambda}^j, u)$ cannot exceed one for all $u \in E_m \cap U_m$ since $\dim E_m > i_m$, $m = 1 \cdots k$. This gives (3.2), so Theorem 3.1 implies boundedness of the $\boldsymbol{\lambda}^j$. Pick $\boldsymbol{\lambda}$ as any limit point, and passing to a subsequence if necessary we asssume $\boldsymbol{\lambda}^j \to \boldsymbol{\lambda}$, $u_m^j \rightharpoonup x_m$. By Lemma 2.4(ii), T_m^j and V_{mn}^j are completely convergent to T_m and V_{mn}, respectively. Theorem 2.5 then gives $\boldsymbol{\lambda} = \boldsymbol{\lambda}^{\mathrm{i}}$, $x = u^{\mathrm{i}}$, and uniqueness of the limit point $\boldsymbol{\lambda}$ comes from Theorem 2.1. Q.E.D.

4. EIGENVECTOR EXPANSIONS

We shall now relate the construction of Corollary 3.3 to that of "limiting" eigentuples used by Atkinson [2, Chap. 11]. The comparison is most direct when $T_m = 0$ in (1.5), e.g., when $\alpha = 0$ in (1.1). Then our conditions on P_m^j coincide with those of [2, Sect. 11.5]. For each j, Atkinson orders the eigenvalues of the V_{mn}^j problem by increasing $\|\boldsymbol{\lambda}\|$, ties being broken by lexicographic ordering of coordinates. "Limiting" eigenvalues are (finite) limit points of the various $\boldsymbol{\lambda}$ sequences as $j \to \infty$, and are eigenvalues of (1.5) [2, Sect. 11.7].

Our result may be interpreted as follows. By ordering the eigenvalues with index $\mathbf{i}$ (for each j) we ensure that each sequence has precisely one (finite) limit point. Thus Atkinson's "limiting eigenvalues" are exactly the $\boldsymbol{\lambda}^{\mathbf{i}}$ for (1.5), and it is in this sense that we sharpen his result. It follows that Atkinson's expansion theorem [2, Theorem 11.10.1] is valid for the $u^{\mathbf{i}}$. To be precise, we need more notation, and we shall revert to a general case, including the T_m.

Let Δ_n, $n = 0 \cdots k$, be the cofactor of ω_n in the operator determinant

$$\Delta = \otimes \begin{vmatrix} \omega_0 & \omega_1 & \cdots & \omega_k \\ T_1 - I_1 & V_{11} & \cdots & V_{1k} \\ \vdots & \vdots & & \vdots \\ T_k - I_k & V_{k1} & \cdots & V_{kk} \end{vmatrix} \tag{4.1}$$

which generates a bounded linear operator on the Hilbert space tensor product $H = \otimes_{m=1}^k H_m$—cf. [7, Chap. 3]. It is known [3, p. 322] that (1.6) implies nonnegative definiteness of Δ_0 on H, so $[x, y]_0 = (x, \Delta_0 y)$ yields an (in general, incomplete) inner product on H. We remark that if x and y are decomposable, say $x = \otimes_{m=1}^k x_m$ and $y = \otimes_{m=1}^k y_m$, then $[x, y]_0 = \delta_0(x, y)$ in our earlier $\oplus_{m=1}^k H_m$ notation. We say that x and y are $[\ ]_0$-orthogonal if $[x, y]_0 = 0$, that the $[\ ]_0$-limit of x_n is zero, written $x_n \to_0 0$, if $\|x_n\|_0 = \Delta_0^{1/2} x_n \to 0$, and so on.

By virtue of Corollary 2.3(ii), we may choose the eigenvectors for the P_m^j-projected problem to be $[\ ]_0$-orthogonal. This orthogonality persists in the limit as $j \to \infty$ (see Corollary 3.3 or [2, Sect. 11.7]) so we have a $[\ ]_0$-orthogonal set of decomposable eigenvectors of H corresponding to $u^{\mathbf{i}} \in U$. Since $\delta_0(x, x) = \det V(x)$, (1.6) allows us to renormalise these eigenvectors and we finally reach $[\ ]_0$-orthonormal decomposable eigenvectors $e^{\mathbf{i}} \in H$, corresponding to $\boldsymbol{\lambda}^{\mathbf{i}}$. The expansion theorem is in terms of these $e^{\mathbf{i}}$.

THEOREM 4.1. *Suppose $f = f_1 \otimes \cdots \otimes f_k, f_m \in H_m$, satisfies*

$$\Delta_n f = \Delta_0 g^n \qquad \text{for some} \quad g^n \in H, \quad n = 1 \cdots k. \tag{4.2}$$

Then f also belongs to the $[\ ]_0$-closure of the span of the $e^{\mathbf{i}}$, and in fact

$$r^{\mathbf{l}} \to_0 0 \qquad \text{as} \quad \min_m l_m \to \infty, \qquad \text{where} \qquad r^{\mathbf{l}} = f - \sum_{i_m = 1}^{l_m} [e^{\mathbf{i}}, f]\, e^{\mathbf{i}}.$$

The arguments of Atkinson [2, Sect. 11.10] carry over with little amendment. The conditions of [2, Theorem 11.10.1] on u appear stronger than ours on f, but (4.2) corresponds to [2, Eq. (11.8.19)] and [2, Eq. (11.8.9)] can be ensured by choosing P_m^1 so that

$$f_m \in \mathscr{R}(P_m^1), \qquad m = 1 \cdots k. \tag{4.3}$$

For a convergence criterion on r^l, Atkinson uses $\|\boldsymbol{\lambda}^j\| \to \infty$ in terms of a single index j. The connection with $\min_m l_m \to \infty$ is provided by Corollary 2.6. With these amendments [2, Theorem 11.10.1] corresponds to the special case $T_m = 0$.

An important improvement stemming from the more specialised analysis of (1.4) rather than (1.5) is that the mode of convergence may be chosen stronger. Specifically, we may choose a new inner product satisfying

$$[x, y]_0' = (x, \Delta_0' y),$$

where

$$\Delta_0' = \bigotimes \det[V_{mn}'].$$

Since

$$\Delta_0 = \Pi \Delta_0' \Pi, \qquad \text{where} \qquad \Pi = \bigotimes_{m=1}^{k} S_m^{-1/2}, \tag{4.4}$$

it is clear that $[\ ]_0$-convergence is weaker $[\ ]_0'$-convergence, at least in infinite dimensional cases.

For a precise statement of the corresponding expansion theorem, we need an analogue of (4.1). An appropriate version of (4.1) is given in, e.g., [3, p. 323]. Briefly, the T_m' are induced into H as self-adjoint operators $T_m^\dagger$ and we write $\mathscr{D}$ for the dense subspace $\bigcap_{m=1}^k \mathscr{D}(T_m^\dagger) \subset H$. Inducing V_{mn}' similarly (as bounded operators $V_{mn}^\dagger$ on H), we define Δ' on $\mathscr{D}$ as for Δ, but with $T_m - I_m$ and V_{mn} replaced by $T_m^\dagger$ and $V_{mn}^\dagger$, respectively. The cofactors Δ_n', $n = 1 \cdots k$, are then defined as symmetric operators on $\mathscr{D}$ and the expansion result is as follows.

COROLLARY 4.2. *Suppose* $f' = f_1' \otimes \cdots \otimes f_k'$, $f_m' \in \mathscr{D}(T_m)$, $m = 1 \cdots k$, *satisfies*

$$\Delta_n' f' = \Delta_0' h^n \qquad \textit{for some} \quad h^n \in \mathscr{R}(\Pi), \qquad n = 1 \cdots k, \tag{4.5}$$

with Π *as in* (4.4). *Then* $f' - \sum_{i_m=1}^{l_m} [\Pi e^{\mathbf{i}}, f']_0' \, \Pi e^{\mathbf{i}} \to_0' 0$ *as* $\min_m l_m \to \infty$.

Proof. By hypothesis $f' \in \mathscr{R}(\Pi^2)$ so we may write $f' = \Pi f$. With $h^n = \Pi g^n$, we easily obtain (4.2), so we may apply Theorem 4.1. The conclusion then follows directly from (4.4). Q.E.D.

This result is of a similar nature to [2, Theorem 11.8.1], with corresponding, but in general incomparable, hypotheses. Atkinson requires bounded invertibility of T'_m, whereas we require it of S_m for $m = 1 \cdots k$. Atkinson essentially requires compactness of $V'_{mn}(T'_m)^{-1}$, whereas we require it of V_{mn} for $1 \leqslant m, n \leqslant k$. Neither set of assumptions contains the other.

The expansibility conditions on f' also overlap between the two results. Our conditions $f'_m \in \mathscr{D}(T_m)$ coincide, while Atkinson uses a weaker version of (4.5), requiring only $h^n \in H$. On the other hand, he restricts the projection scheme in various ways [2, Eqs. (11.5.2), (11.5.4)]. Thus the condition [2, Eq. (11.8.9)]

$$P^j_m f'_m = f'_m, \qquad j = 1, 2, \ldots, \tag{4.6}$$

restricts f' in [2, Theorem 11.8.1]. In Corollary 4.2, however, (4.6) follows from (4.3) and hence is no restriction on f'. In [6], a similar assertion is made, but the argument there is invalid. Specifically, [6, Theorems 5.1 and 5.3] should include the analogue [6, Eq. (4.3)] of (4.6). Incidentally Volkmer [8] has recently derived an expansion for arbitrary elements $f' \in H$, at the price of stronger (e.g., commutativity) conditions on the V'_{mn}.

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References

1. F. M. Arscott, "Periodic Differential Equations: An Introduction to Mathieu, Lamé, and Allied Functions," Macmillan, New York, 1964.
2. F. V. Atkinson, "Multiparameter Eigenvalue Problems," Vol. 1, Academic Press, New York, 1972.
3. P. A. Binding, Multiparameter definiteness conditions, *Proc. Roy. Soc. Edinburgh Sect. A* **89** (1981), 319–332.
4. P. A. Binding, Dual variational approaches to multiparameter eigenvalue problems, *J. Math. Anal. Appl.*, in press.
5. P. A. Binding and P. J. Browne, A variational approach to multiparameter eigenvalue problems in Hilbert space, *SIAM J. Math. Anal.* **9** (1978), 1054–1067.
6. P. A. Binding, A. Källström, and B. D. Sleeman, An abstract multiparameter spectral theory, *Proc. Roy. Soc. Edinburgh Sect. A* **92** (1982), 193–204.
7. B. D. Sleeman, "Multiparameter Spectral Theory in Hilbert Space," Pitman, London, 1978.
8. H. Volkmer, On the completeness of eigenvectors of right definite multiparameter problems, *Proc. Roy. Soc. Edinburgh Sect. A*, in press.